Groupoids in combinatorics – applications of a theory of local symmetries

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Abstract

An objective of the theory of *combinatorial groupoids* is to introduce concepts like "holonomy", "parallel transport", "bundles", "combinatorial curvature" etc. in the context of simplicial (polyhedral) complexes, posets, graphs, polytopes, arrangements and other combinatorial objects. In this paper we give an exposition of some of the currently most active research themes in this area, offer a unified point of view, and provide a list of prospective applications in other fields together with a collection of related open problems.

1 Introduction

This paper is a sequel to [Ž05] where a program for developing a theory of combinatorial groupoids was originally outlined. The main objective of [Ž05] was to demonstrate the relevance of this theory for some well known problems of contemporary geometric combinatorics, notably the graph coloring problem (Lovász conjecture and its relatives) and the problems related to cubifications of manifolds.

In this paper we offer a broader perspective on this subject. More general concepts are introduced, both old and new applications are discussed or at least outlined and, what is potentially the most important aspect of the paper, we try to collect together other related developments where combinatorial groupoids were implicitly or explicitly used.

Hoping that this paper may serve as an invitation to the subject, we included a large number of examples of problems of combinatorial nature, among them the Penrose impossible "tribar" and the S. Lloyd "15 game", where the ideas and the techniques of the theory of groupoids may play an important role.

1.1 An overview

Recent publications [BGH] [J01] [Ž05] offer a quite convincing evidence that the language and methods of the theory of groupoids, after being successfully applied in other major mathematical fields, offer new insights and perspectives for applications in combinatorics and discrete and computational geometry.

The groupoids (groups of projectivities) have recently appeared in geometric combinatorics in the work of M. Joswig [J01], see also a related paper with Izmestiev [IJ02] and the references to these papers, where they have been applied to toric manifolds, branched coverings over S^3, colorings of simple polytopes etc.

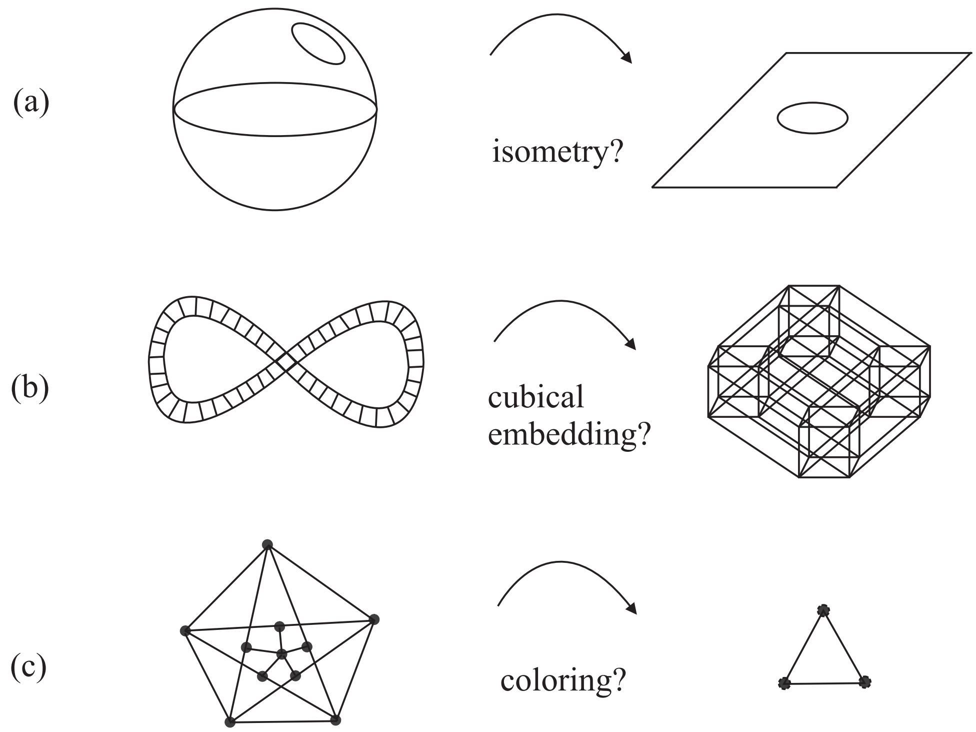


Figure 1: Is there a common point of view!?

Even more explicitly the concepts of "connection", "geodesics", "holonomy" have appeared in [BGH]. Motivated by the theory of GKM-manifolds (named after Goresky, Kottwitz, and MacPherson [GKM]) Bolker, Guillemin, and Holm develop in this paper an analogy between graph theory and the theory of manifolds.

The main purpose of [Ž05] was to show that these and related developments should not be seen as isolated examples. Quite the opposite, they serve as a motivation for further extensions and generalizations and call for a systematic applications of a theory of local symmetries in combinatorics.

As a first application it was shown in [Ž05] that a cubical analogue of Joswig's groupoid provides new insight in cubical complexes non-embeddable into cubical lattices (a question related to a problem of S.P. Novikov which arose in connection with the 3-dimensional Ising model) [BP02] [N96]. The second, perhaps more far reaching application developed in this paper, was a generalization, both to more general test graphs and to simplicial complexes, of a recent resolution of the Lovász conjecture by Babson and Kozlov [BK04].

It appears that combinatorial groupoids are hidden in the background of many contemporary combinatorial constructions and applications. It is

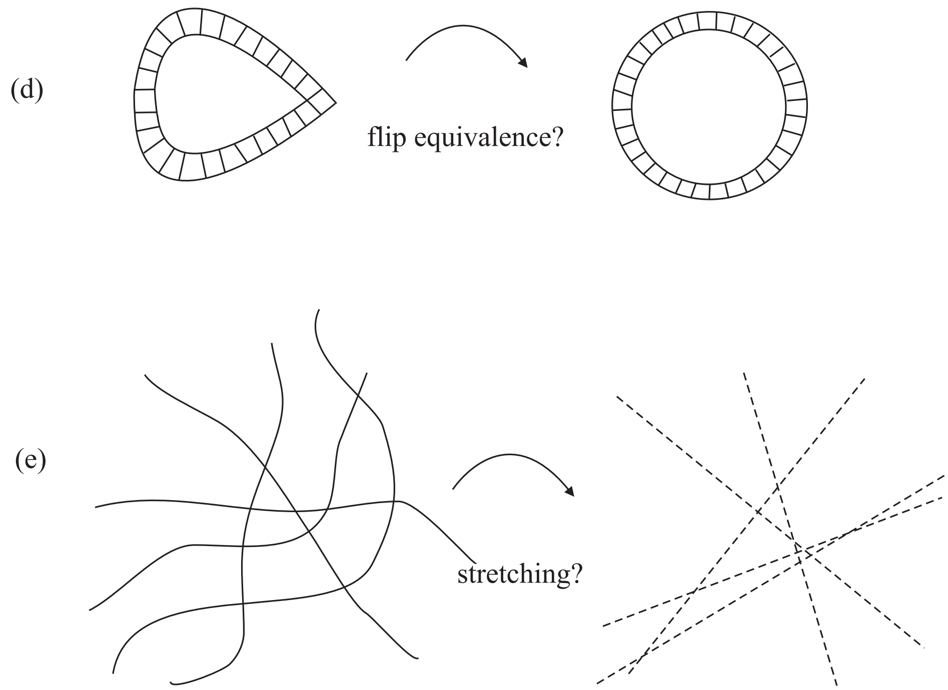


Figure 2: Groupoids provide a theory of local symmetries.

potentially a very interesting project to analyze the role of groupoids in the papers like [BBLL] [BKLW] [BL] [Ga04] etc. One of our central objectives in this paper is to advocate a systematic use of groupoids as a valuable tool for geometric and algebraic combinatorics.

1.2 The first unifying theme

Our point of departure is an observation that different problems from different mathematical disciplines, in particular some well known problems of combinatorial geometric nature, can be all approached from a similar point of view.

The unifying theme and a single point of view is provided by the concept of a groupoid. The reader is referred to Figures 1, 2 and 3 for an informal list of questions which all seem to involve a concept of a groupoid.

In each of the listed cases there is or ought to be a groupoid naturally associated to an object of the given category. For each of these groupoids there is an associated "parallel transport", holonomy groups and other related invariants which serve as obstructions for the existence of morphisms indicated in Figure 1.

If $M = (M, g)$ is a Riemannian manifold, the associated groupoid $\mathcal{G} = \mathcal{G}_M$ has M as the set of objects while the morphism set $\mathcal{G}(x, y)$ consists of all linear isomorphisms $a : T_xM \to T_yM$ arising from a parallel transport along piece-wise smooth curves from x to y. One of the manifestations of Gauss "Theorema Egregium" is that the associated holonomy group is a metric

(f)

(g)

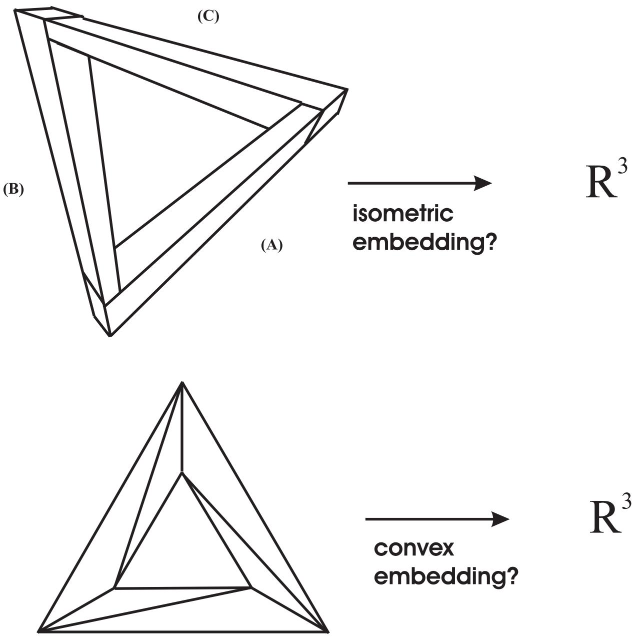


Figure 3: Locally possible may be globally impossible!

invariant, consequently an isometry depicted in Figure 1 (a) is not possible.

In a similar fashion a cubical “Theorema Egregium” [Ž05, Theorem 3.7] provides an obstruction for an embedding of a cubical complex into a hypercube or a cubical lattice. As a consequence the cubification (quadrangulation) of the ∞-shaped complex depicted in Figure 1 (b) does not admit a cubical embedding into a hypercube for the same formal reason (incompatible holonomy) a spherical cap cannot be isometrically represented in the plane.

Perhaps it comes as a surprise that the graph coloring problem, Figure 1 (c), can be also approached from a similar point of view. An analysis of holonomies (“parallel transport”) of diagrams of Hom-complexes of graphs (simplicial complexes) over the associated Joswig’s groupoid (Example 2.4) eventually leads to a general result [Ž05, Theorem 4.21] which includes the “odd” case of Babson-Kozlov-Lovász coloring theorem as a special case.

1.3 The second unifying theme

The problems depicted in Figures 1, 2 and 3 can all be seen as instances of the following general problem-scheme.

Problem 1: [Z92] Given some kind of (combinatorial) structure $\Re$, is it always possible to embed it into a very “regular” or “complete” structure of this kind. Alternatively and perhaps more generally, a “regular” or “complete” structure may be replaced by some other kind of environment (space) inhabited by structures similar to $\Re$.

G. Ziegler in [Z92] provides a list of combinatorial problems which can be placed in this category. They are all of distinct combinatorial flavor. For

example the first question is whether each matroid of rank 3 can be embedded into a finite projective plane, while the second ask if each *Steiner triple system* can be embedded into a finite *Kirkman system* (resolvable Steiner system).

Ziegler makes an interesting remark in this paper to the effect that "there should exist cohomology theories that can handle these embedding problems". For an interesting evidence that such "non-classical applications of cohomology theory to embedding problems" should exist, he refers the reader to [Pe] and [CrRy].

One can speculate that combinatorial groupoids may provide such "cohomology theory" in some favorable situations. Indeed, the essence of the classical Chern-Weyl theory is the construction of characteristic cohomology classes from the curvature of a manifold and the curvature is just a manifestation of the holonomy phenomenon.

For example in the simplest possible situation, an "obstruction cocycle" evaluated on a 2-dimensional cell, measures the holonomy around this cell. In other words the information usually captured by cohomology often comes from the groupoid (connection, holonomy) naturally associated to the problem.

It is plausible that in majority of "embedding problems" listed in [Z92], similarly in each of problems symbolically depicted in Figures 1–3, one should be able to identify combinatorial groupoids which are naturally associated to these objects (complexes, graphs, matroids, triple systems, configurations, arrangement etc.).

An embedding induces a morphism of groupoids (often of a very special type, which implies a monomorphism on the level of holonomy groups). Already this yields non-embeddability in some cases (e.g. examples of cubical complexes which cannot be embedded in cubical lattices, Figure 1 (b)).

However, a cohomology theory we are after ought to be much more subtle instrument for proving non-embaddibility.

It is fascinating that such a scheme already exists in some sense, once we identify the groupoids and use objects (matroids, triple systems, configurations, arrangement) to define natural bundles over these groupoids.

We pass from groupoids to the associated convolution algebras (in the same manner one goes from a group to the group algebra or from a poset to its incidence algebra) and interpret the natural bundles as moduli over these algebras. After that we are in the situation which is pregnant with possibilities!

As already emphasized, the classical Chern-Weyl theory is the construction of characteristic cohomology classes from the curvature of a manifold, and the relevant information about the curvature is captured by the underlying groupoid. Today this construction is incorporated into a map from K-theory of an algebra (say convolution algebra of a groupoid) to the cyclic homology of the algebra (Connes, Karoubi etc.).

This is a recipe which in the case of combinatorial groupoids should be quite concrete and this appears to be a good candidate for an adequate cohomology theory!

2 Groupoids

Groups and symmetries have been treated almost as synonyms in the history of mathematics. Indeed, we have all been trained that wherever we encounter symmetries, there ought to be a group of transformations in the background. Consequently it may come as a surprise that the concept of a group is sometimes not sufficient to deal with this phenomenon in general. Indeed, it may not be widely known that not groups but *groupoids* allow us to handle objects which exhibit what is clearly recognized as symmetry although they admit no global automorphism whatsoever. Unlike groups, groupoids are capable of describing reversible processes which can pass through a number of states. For example according to A. Connes [C95], Heisenberg discovered quantum mechanics by considering the groupoid of quantum transitions rather than the group of symmetry.

Groupoids are formally defined as small categories $\mathcal{C} = (Ob(\mathcal{C}), Mor(\mathcal{C}))$ such that each morphism $\alpha \in Mor(\mathcal{C}))$ is an isomorphism. This condition guarantees that each process governed by a groupoid is reversible. The reader is referred to [Br] [Br87] [Br97] [H71] [W96] for expositions of different aspects of the theory of groupoids. The *vertex* (isotropy) group $\Pi(\mathcal{C}, x) := \mathcal{C}(x, x)$ is often referred to as the *holonomy* group of $\mathcal{C}$ at $x \in Ob(\mathcal{C})$. A very sim-

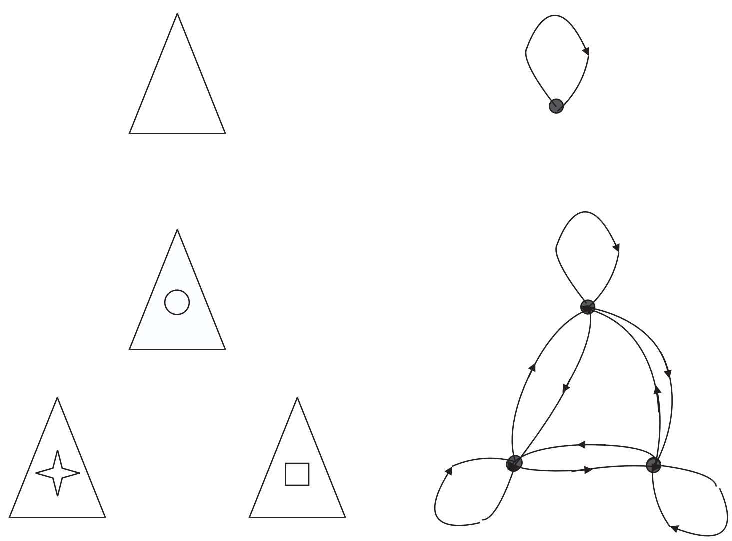


Figure 4: From groups to groupoids.

ple example of a groupoid, which nevertheless exposes some of its interesting features, is given in Figure 4. The isosceles triangle has only one non-trivial symmetry, hence its group of symmetries is $\mathbb{Z}_2$. This fact is conveniently recorded in the directed graph (digraph) depicted on the right of this triangle. It has only one vertex, corresponding to one object, or one state of the object, and a directed loop associated to the non-trivial automorphism.

What if the triangle can pass through a number of different states, say change its color or if some changing geometric pattern is present on the trian-

gle? Each "state" of the object is associated a different node in the graph while arrows and directed loops record all possible transformations between states. The directed graph obtained this way is precisely the associated groupoid if we agree that the arrows (transformations) can be composed, provided they are "composable" in the sense that $\alpha \circ \beta$ exists only if the source object of α coincides with the target object of β.

Example 2.1. An excellent example of how groupoids may appear in combinatorial practise arises from the analysis of the local symmetries associated to Penrose tribar, Figure 3 (f). One of the guiding principles of [Pe] is that this figure is "locally consistent" i.e. by covering (removing) one side of the tribar, the remaining two sides are unambiguously visualized in the surrounding 3-space. Moreover, assuming that all three sides are mutually congruent parallelepipeds, given two of them, say (A) and (B) in Figure 3, there is a natural isometry (local symmetry) $\alpha_{BA} : (A) \to (B)$ sending one parallelepiped to another. In order to make description of α_{BA} easier to follow, the parallelepipeds are depicted in Figure 3 so that one side appears to be thinner than the other. It is assumed that α_{BA} maps the thinner side of (A) to the thinner side of (B). The local symmetries $\alpha_{CB} : (B) \to (C)$ and $\alpha_{AC} : (C) \to (A)$ are similarly defined. It turns out that

$$\alpha_{AC} \circ \alpha_{CB} \circ \alpha_{BA} =: \eta_A$$

is not an identity map. Rather, it is a rotation of the parallelepiped (A) through the angle of $90°$ around the longest axes of symmetry.

We conclude that a groupoid associated to the Penrose tribar has three objects, (A), (B) and (C), and the holonomy group $\Pi \cong \mathbb{Z}_4$. In other words, the corresponding digraph is very similar to the digraph depicted in Figure 4.

2.1 Generalities about "bundles" and "parallel transport"

The notion of a groupoid is a common generalization of the concepts of space and group, i.e. the theory of groupoids allows us to treat spaces, groups and objects associated to them from the same point of view. In the case of spaces this is achieved by associating a "path groupoid" or the fundamental groupoid πX to a space X, [Br, Chapter 6]. Given a group G and a G-set S, the associated groupoid has S as a set of objects while morphisms are all "arrows" $x \xrightarrow{g} y$ such that $x, y \in S, g \in G$ and $gx = y$. In particular G itself is a category with only one object and G as the set of morphisms.

A common generalization for the concept of a bundle Y over X and a G-space Y is a $\mathcal{C}$-space or more formally a diagram over the groupoid $\mathcal{C}$ defined as a functor $F : \mathcal{C} \to Top$.

The reader is referred to [Br] [Br87] [Br97] [W96] and the references in these sources for more information about groupoids.

Here we provide only a list (glossary) of some of the basic concepts associated to groupoids in the form that will allow their immediate use in subsequent sections.

GLOSSARY

> **Groupoid:** A small category $\mathcal{C} = (O, M)$ where $O = Ob(\mathcal{C})$ is the set of objects and $M = Mor(\mathcal{C})$ the set of morphisms. Informally speaking the groupoid $\mathcal{C}$ provides a "road map" on O which can be visualized as a digraph as in Figure 4. The vertex (holonomy) group at $x \in O$ is $\Pi_x := Hom_{\mathcal{C}}(x, x)$. If $\mathcal{C}$ is connected all its vertex groups are isomorphic and often denoted by $\Pi\mathcal{C}$ or Π.
>
> **Bundle over O:** A collection $\mathcal{X} = \{X_i\}_{i\in O}$ of spaces or sets (fibres) labelled (indexed) by elements of the set O. A bundle often arises from a map $f : X \to O$ with $X_i := f^{-1}(i)$ as the fibre over $i \in O$.
>
> **Connection on $\mathcal{X}$:** A "connection" or "parallel transport" on the bundle $\mathcal{X} = \{X(i)\}_{i\in O}$ is a functor (diagram) $\mathcal{P} : \mathcal{C} \to Top$ such that $X(i) = \mathcal{P}(i)$ for each $i \in S$. Informally speaking, the groupoid $\mathcal{C}$ provides a "road map" on S, while the functor $\mathcal{P}$ defines the associated transport from one fibre to another.

It follows from these definitions that a bundle $\mathcal{X}$ is just a map $O \to Top$ while a connection extends this map to a functor $\mathcal{C} \to Top$.

2.2 Principal bundles associated to a groupoid

There are several "tautological" bundles associated to a groupoid $\mathcal{C} = (O, M)$. For example one can associate to $x \in O$ the vertex group $\Pi_x = Hom_{\mathcal{C}}(x, x)$. There is a natural connection $\mathcal{P} : \mathcal{C} \to Set$ on this bundle where for $\alpha \in Hom(x, y)$, $\mathcal{P}(\alpha) : \Pi_x \to \Pi_y$ is defined by $\mathcal{P}(\alpha)(\beta) := \alpha \circ \beta \circ \alpha^{-1}$.

The concept of a principal or "frame" bundle seems to be of equal importance in applications of (combinatorial) groupoids. This notion is a natural unification of the concept of a principal bundle over a topological space and a free G-set.

Definition 2.2. *Suppose that O is a set and assume that $\mathcal{C} = (O, C)$ and $\mathcal{D} = (O, D)$ are two groupoids on O as the set of objects. Moreover, assume that $\mathcal{C}$ is a subgroupoid of $\mathcal{D}$ in the sense that $C \subset D$ and that $\mathcal{D}$ is connected as a groupoid. Given an object $a \in O$, define a bundle $Fr = Fr_a : O \to Set$ by the formula $Fr_a(y) := \mathcal{D}(a, y)$. This bundle naturally comes with both a $\mathcal{D}$ and $\mathcal{C}$-connection. The isomorphism type of this bundle (connection) is independent of the choice of object a (as a consequence of connectedness of $\mathcal{D}$). This bundle together with the associated $\mathcal{C}$-connection is referred to as a $(\mathcal{C}, \mathcal{D})$-principal (or frame) bundle over O.*

Usually it is the groupoid $\mathcal{C}$ we are interested in. The auxiliary groupoid $\mathcal{D}$ often appears as a natural "ambient" groupoid for $\mathcal{C}$. For example if $\mathcal{C}$ is a free G-set, then $\mathcal{D}$ is the groupoid associated to the set $S = Ob(\mathcal{C})$ as a Q-set where $Q \supseteq G$ is the group of all permutations of S.

If $\mathcal{C} = \mathcal{G}_M$ is the groupoid associated to a Riemannian manifold (M, g), described in Section 1.2, then $\mathcal{D}$ is the groupoid $Vect_M$ which associates to a pair

of points (objects) (x, y) in M, the morphism set $\mathcal{D}(x, y) = Vect(T_xM, T_yM)$ of all linear isomorphisms from T_xM to T_yM. This is the reason why $Fr = Fr_{\mathcal{C}}$ is also referred to as a frame bundle, since in this case $Vect(\mathbb{R}^n, T_xM)$ is the set (manifold) of all n-frames in T_xM.

This situation arises in all cases where objects of the groupoid have natural *external isomorphisms*, in particular the group $Hom_{\mathcal{D}}(x, x)$ of external isomorphisms of x may be larger than $\Pi_x = Hom_{\mathcal{C}}(x, x)$. This is clearly the case with the groupoid $\mathcal{G}_M$ where the natural group of symmetries of T_xM is isomorphic to $GL(n, \mathbb{R})$.

All combinatorial groupoids discussed in Sections 2.3 and 2.4 are of this kind. In all these examples the natural (external) isomorphisms are structure preserving bijections associated to these objects. In the case of the Joswig groupoid $J(K)$, the natural isomorphisms are bijective simplicial maps of d-simplices so the (external) symmetry group of a d-simplex is the group of all permutations of its vertices, isomorphic to S_{d+1}. In the case of groupoids associated to games the situation is similar. The external group of symmetries of a position of a game (Section 2.3) is the group of all permutations of the pieces, e.g. in the "15 game" it is the group S_{15}. The external group of symmetries arising in the context of pure d-dimensional, cubical complexes is the group B_d of symmetries of a d-cube etc..

In all these examples there is a tautological "outer groupoid" $\mathcal{D}$ and the associated frame bundle $Fr_{\mathcal{C}}$.

Symmetry breaking patterns: Suppose that $\mathcal{C}$ is a groupoid where all objects $x \in Ob(\mathcal{C})$ are mutually *externally isomorphic*, i.e. isomorphic from the point of view of their inner (combinatorial or geometric) structure. As a consequence there is a natural "external", connected groupoid $\mathcal{D}$ associated to $\mathcal{C}$ such that $\mathcal{C}$ is a subgroupoid of $\mathcal{D}$. In other words $Ob(\mathcal{D}) = Ob(\mathcal{C})$ while morphisms in $\mathcal{D}$ are external morphisms. Let $Fr = Fr_a$ be the associated $(\mathcal{C}, \mathcal{D})$-frame bundle. An element of $Fr(x) = Hom_{\mathcal{D}}(a, x)$ is interpreted as a "symmetry breaking pattern" on x. Examples of such patterns are exhibited in Figures 6 and 7 and they are a useful bookkeeping device for keeping track of the holonomies, for the concrete combinatorial description of the associated covering groupoids etc.

The standard question: Given $\mathcal{C}$ and the associated "outer" groupoid $\mathcal{D}$, it is interesting to know whether the associated point groups are different, i.e. if $Hom_{\mathcal{C}}(x, x) \subsetneq Hom_{\mathcal{D}}(x, x)$.

2.3 Combinatorial groupoids; first examples

Combinatorial groupoids are the groupoids that appear in combinatorics. This is certainly not a very informative statement so we offer a few examples for illustrative purposes. More formal definitions are offered in Section 2.3.

Suppose that $\mathcal{G}$ is some kind of a "game" played on a "board" $\mathcal{B}$ with some "pieces" $\mathcal{P}$ that can be moved around this board according to some "rules" $\mathcal{R}$. It can be a one-player game, for example a game with playing cards as pieces

(the game of "Solitaire" is an example), a two-player game (chess, checkers etc.) or a multiplayer game (e.g. some multiplayer computer game). We will ignore here the "dynamical" aspect of the game and focus on the "states" (positions) of the game $\mathcal{G}$ and how one can, according to the rules of the game, move from one state to another.

In order to have a concrete example before our eyes, let us assume that the board $\mathcal{B}$ is a $(m \times n)$-chessboard and that pieces cannot be distinguished from one another, like in the game of checkers. A distribution of pieces on the board $\mathcal{B}$ is called a position (state) of the game. One can pass from one position to another by rearranging one or more pieces, i.e. once the game is started pieces are neither removed from nor returned to the board. An important aspect of this type of the game is that it is *reversible* i.e. we can always return to the original position of pieces by performing the inverse moves. As customary

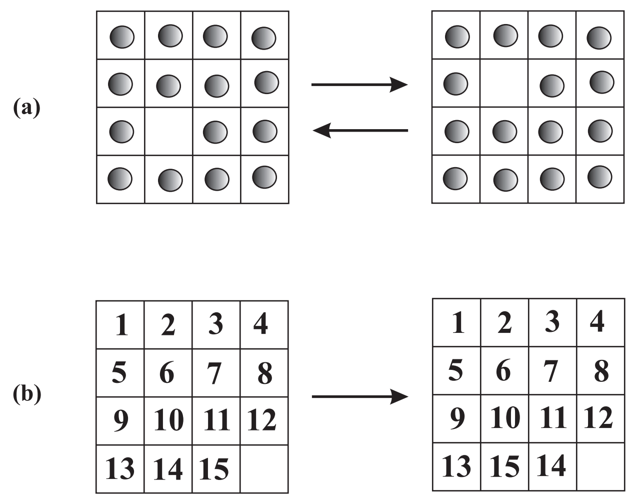


Figure 5: The Lloyd "15 game".

in game theory, one can associate a directed graph $D(\mathcal{G}) = (N(\mathcal{G}), E(\mathcal{G}))$ to the game $\mathcal{G}$. The nodes $N(\mathcal{G})$ are all allowed positions of the game and pairs (p, q) of positions form a directed edge in $E(\mathcal{G})$ if and only if the rules $\mathcal{R}$ allow a move from position p to position q. It is clear that this directed graph is actually a groupoid.

It is often convenient to encode all possible positions of the game in a simplicial complex $K(\mathcal{G})$. The vertices of $K(\mathcal{G})$ are all elementary cells (i, j) of the board $\mathcal{B} = [m] \times [n]$ and each position $p \in \mathcal{P}$ of the game contributes a (maximal) simplex $\sigma = \sigma(p)$, where $(i, j) \in \sigma$ if and only if the cell $(i, j) \in [m] \times [n]$ is occupied by a piece.

Conversely, given a pure simplicial complex K, one can interpret its maximal simplices as the set of all allowed positions of a game $\mathcal{G}(K)$ which is played on the set $V(K)$ of vertices of K. Moreover, assume that there is only one rule that specifies that one can change positions by moving only one piece at a time. The groupoid arising this way is precisely the Joswig groupoid $\mathcal{J}(K)$ of K (Section 3.1)!

One can have even more restrictive rule by asking that only pieces that satisfy some other constraint can be moved to another position from a list of allowable positions. A perfect example of such a game is the famous "15 game" of Samuel Lloyd, "America's greatest puzzle creator", see `http://www.holotronix.com/samlloyd15a.html`. In this game 15 pieces are placed on a (4×4)-chessboard, Figure 5 (a), and a piece can be moved only if it is an immediate neighbor of the unoccupied cell. Let $Lloyd_{15} = (O, M)$ be the associated groupoid, i.e. the objects of this groupoid are all 16 ways to put 15 identical pieces on a (4×4)-chessboard and morphisms are moves allowed by the "15 game".

The famous Lloyd's "15–14" problem is to start with 15 labelled pieces, positioned as in Figure 5 (b) on the left and, playing the "15 game", end up in the position depicted in Figure 5 (b) on the right. It turns out that this is not possible. We see this fact as a manifestation of a phenomenon that

$$\Pi(Lloyd_{15}) = A_{15} \subsetneqq S_{15} \tag{1}$$

i.e. that the holonomy group of the groupoid $Lloyd_{15}$ is different from the a priori given group of symmetries of the object!

2.4 Combinatorial groupoids; general picture

In this section we introduce a sufficiently general class of combinatorial groupoids which seems to capture the essential features of all examples reviewed in this paper. We warn the reader that this is certainly not the most general framework suitable for all possible applications. Rather, as emphasized in [Ž05], we create "an ecological niche for combinatorial groupoids which may be populated by new examples and variations as the theory develops".

Definition 2.3. *Suppose that $(P, \leq)$ is a (not necessarily finite) poset. Suppose that Σ and Δ two families of subposets of P. Choose $\sigma_1, \sigma_2 \in \Sigma$. If for some $\delta \in \Delta$ both $\delta \subset \sigma_1$ and $\delta \subset \sigma_2$, then the posets σ_1, σ_2 are called δ-adjacent, or just adjacent if δ is not specified. Define $\mathcal{C} = (Ob(\mathcal{C}), Mor(\mathcal{C}))$ as a small category over $Ob(\mathcal{C}) = \Sigma$ as the set of objects as follows. For two δ-adjacent objects σ_1 and σ_2, an* elementary morphism $\alpha \in \mathcal{C}(\sigma_1, \sigma_2)$ *is an isomorphism $\alpha : \sigma_1 \to \sigma_2$ of posets which leaves δ point-wise fixed. A morphism $\mathfrak{p} \in \mathcal{C}(\sigma_0, \sigma_m)$ from σ_0 to σ_m is an isomorphism of posets σ_0 and σ_m which can be expressed as a composition of elementary morphisms.*

Given two adjacent objects σ_1 and σ_2, an elementary morphism $\alpha \in \mathcal{C}(\sigma_1, \sigma_2)$ may not exist at all, or if it exists it may not be unique. In case it exists and is unique it will be frequently denoted by $\overrightarrow{\sigma_1\sigma_2}$ and sometimes referred to as a "flip" from σ_1 to σ_2. In this case a morphism $\mathfrak{p} \in \mathcal{C}(\sigma_0, \sigma_m)$ is by definition a composition of flips

$$\mathfrak{p} = \overrightarrow{\sigma_0\sigma_1} * \overrightarrow{\sigma_1\sigma_2} * \ldots * \overrightarrow{\sigma_{n-1}\sigma_n}. \tag{2}$$

Caveat: Here we adopt a useful convention that $(x)(f * g) = (g \circ f)(x)$ for each two composable maps f and g. The notation $f * g$ is often given priority

over the usual $g \circ f$ if we want to emphasize that the functions act on the points from the right, that is if the arrows in the associated formulas point from left to the right.

Suppose that P is a ranked poset of depth n with the associated rank function $r : P \to [n]$. Let $\mathcal{E} = \mathcal{E}_P$ be the $\mathcal{C}$-groupoid described in Definition 2.3 associated to the families $\Sigma := \{P_{\leq x} \mid r(x) = n\}$ and $\Delta := \{P_{\leq y} \mid r(y) = n-1\}$. It is clear that other "rank selected" groupoids can be similarly defined.

The definitions of groupoids $\mathcal{C}$ and $\mathcal{E}$ are easily extended from posets to simplicial, polyhedral, or other classes of cell complexes. If K is a complex and $P := P_K$ the associated face poset, then $\mathcal{C}_K$ and $\mathcal{E}_K = \mathcal{E}(K)$ are groupoids associated to the poset P_K. We will usually drop the subscript whenever it is clear from the context what is the ambient poset P or complex K.

Example 2.4. Suppose that K is a pure, d-dimensional simplicial complex. Let $\mathcal{E}(K)$ be the associated $\mathcal{E}$-groupoid corresponding to ranks d and $d-1$. Then the *groups of projectivities* $\Pi(K, \sigma)$, introduced by Joswig in [J01], are nothing but the holonomy groups of the groupoid $\mathcal{E}(K)$. For this reason the groupoid $\mathcal{E}(K)$ is in the sequel often referred to as Joswig's groupoid and denoted by $\mathcal{J}(K)$. $\mathcal{J}(K)$ is *connected* as a groupoid if and only if K is "strongly connected" in the sense of [J01].

A simplicial map of simplicial complexes is non-degenerate if it is 1–1 on simplices. The following definition extends this concept to the case of posets.

Definition 2.5. *A monotone map of posets $f : P \to Q$ is non-degenerate if the restriction of f on $P_{\leq x}$ induces an isomorphism of posets $P_{\leq x}$ and $Q_{\leq f(x)}$ for each element $x \in P$. Similarly, a map of simplicial, cubical or more general cell complexes is non-degenerate if the associated map of posets is non-degenerate. In this case we say that P is* mappable *to Q while a non-degenerate map $f : P \to Q$ is often referred to as a* combinatorial immersion *from P to Q.*

Example 2.6. A graph homomorphism [Ko] $f : G_1 \to G_2$ can be defined as a non-degenerate map of associated 1-dimensional cell complexes. A n-coloring of a graph G is a non-degenerate map (graph homomorphism) $f : G \to K_n$ where K_n is a complete graph on n vertices.

Proposition 2.7. *Suppose that P and Q are ranked posets of depth n and let $f : P \to Q$ be a non-degenerate map. Then there is an induced map (functor) $F : \mathcal{E}_P \to \mathcal{E}_Q$ of the associated $\mathcal{E}$-groupoids. Moreover, F induces an inclusion map $\Pi(\mathcal{E}_P, \mathfrak{p}) \hookrightarrow \Pi(\mathcal{E}_Q, F(\mathfrak{p}))$ of the associated holonomy groups.*

3 Applications

3.1 Joswig groupoid $\mathcal{J}(K)$

M. Joswig defined parallel transport and the associated holonomy groups in the context of each pure, d-dimensional simplicial complex K in [J01]. He did

not formally use the language of the theory of groupoids, but the combinatorial groupoids are implicit in this and subsequent paper [J01b], and in the joint paper with I. Izmestiev [IJ02]. In particular, our definition of combinatorial groupoids and associated concepts is strongly influenced by Joswig's point of view and reflects a desire to incorporate other examples of apparently similar nature into the same framework.

Formally speaking, the Joswig groupoid is the $\mathcal{E}$-groupoid (Section 2.4) associated to a pure d-dimensional simplicial complex K (Example 2.4).

More explicitly the objects of $\mathcal{K}$ are d-dimensional simplices of K while the morphisms are compositions of "flips" (Figure 6), as in the equation (2), Section 2.4.

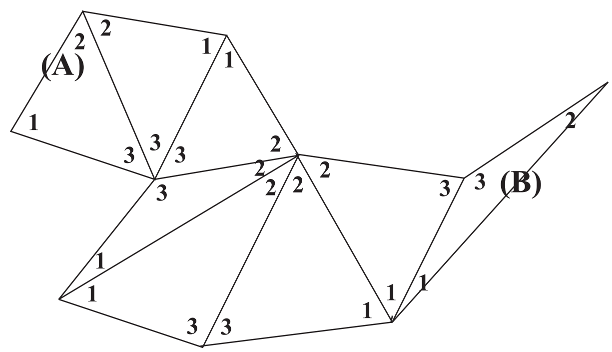


Figure 6: Parallel transport from **(A) to (B).**

A "symmetry braking pattern", in the sense of Section 2.2, which can be used for keeping the track of the action of a Joswig groupoid is simply a labelling of vertices of the corresponding simplex, see e.g. simplex (A) in Figure 6. The holonomy group is a priory a subgroup of the group of all permutations of vertices of a simplex, i.e. a subgroup of the group S_{d+1}.

Let us briefly review the original problem that motivated M. Joswig to introduce his "groups of projectivities", i.e. the holonomy groups of the groupoid $\mathcal{J}(K)$.

M. Davis and T. Januszkiewicz associated a smooth $(n+d)$-dimensional manifold $\mathcal{Z}_P$ to any d-dimensional, simple convex polytope P with n facets. These are examples of quasi-toric manifolds [BP02], relatives of toric varieties, which come equipped with a T^n-action. Genuine toric varieties are of dimension $2d \leq n+d$. Perhaps motivated in part by this, V. Buchstaber suggested a program of studying when one can find a subgroup T of T^n which acts freely on $\mathcal{Z}_P$; in that case $\mathcal{Z}_P/T$ would be another quasi-toric manifold of dimension $n+d-\dim(T)$. Let $s(P)$ is the maximum dimension of a subgroup $T \subseteq T^n$, acting freely on $\mathcal{Z}_P$. I. Izmestiev in [I01] defined the chromatic number $\gamma(P)$ of P as the minimal number of colors required to color the facets of P such that any two facets sharing a vertex have different colors. The relation between $s(P)$ and $\gamma(P)$ is given by inequalities

$$n - \gamma(P) \leq s(P) \leq n - d$$

where the right hand inequality is elementary while the left hand relation is due to Izmestiev [I01].

One of the consequences of the Joswig's analysis of holonomy groups of the groupoid $\mathcal{J}(K)$ where K is the dual of P, is the result [J01, Theorem 16] which implies that $\gamma(P) = d$ if the corresponding holonomy group is trivial.

The reader is referred to [I01] [I01b] [IJ02] [J01] [J01b] for these and other applications of group of projectivities of simplicial complexes.

3.2 How is graph like a manifold?

One of the central ideas of [BGH] is to approach classical combinatorial problems by exploring analogies between graphs and manifolds. A central theme, illuminating this connection, arises from the theory of group actions, notably from the analysis of complex $(\mathbb{C}^*)^n$-manifolds. The class of so called GKM-manifolds, named after Goresky, Kottwitz, and MacPherson, has a particularly interesting structure theory. More precisely, the 0-dimensional orbits, as nodes, and 1-dimensional orbits, as edges, define an associated GKM-graph $\Gamma = \Gamma(M)$ which captures a substantial part of the structure of the original GKM-manifold M. The graph Γ arising this way turns out to be d-regular, where d is the dimension of the underlying complex $\mathbb{C}^n$-manifold M. An extra piece of structure is an assignment of integer vectors (axial functions) to edges of this graph, which taken together define an "embedding" of the graph in $\mathbb{R}^n$.

Bolker, Guillemin, and Holm, building on the previous work of Goresky, Kottwitz, MacPherson, Rosu, Knutson, Lian, Liu, Yau, Zara, and others, develop in this paper a dictionary associating manifold concepts to graph concepts.

A particularly interesting aspect of this work is appearance of *connections, holonomy groups, geodesics*, etc. in the context of arbitrary (regular) graphs $\Gamma = (V, N)$[1]. Here is one of the main definitions.

Definition 3.1. [BGH] *A* connection *on a graph $\Gamma = (V, E)$ is a collection of bijective functions $\nabla_{(x,y)} : Star(x) \to Star(y)$, indexed by all (oriented!) edges (x, y) in Γ, where $Star(z) := \{(z, w) \in E \mid w \in V\}$ is the set all oriented edges in Γ incident to z. These functions satisfy the following conditions:*

(1) $\nabla_{(x,y)}(x, y) = (y, x)$,

(2) $\nabla_{(y,x)} = \nabla^{-1}_{(x,y)}$.

Bolker, Guillemin, and Holm use the connection ∇ to define geodesics in the graph Γ, to introduce its totally geodesic subgraphs, holonomy groups as subgroups of the groups of all permutations of $Star(x)$ etc., see [BGH] for the detailed development and applications of these concepts.

[1] I am grateful to M. Joswig for drawing my attention to this fact!

The reader is invited to identify the associated combinatorial groupoid $\mathcal{G}_\Gamma$ and to relate it to the groupoids described in Sections 2.3 and 2.4. The associated bundle where this connection (parallel transport) is defined) is clearly the collection $\{Star(x)\}_{x\in V}$.

Answering the question from the title of their paper (and our Section 3.2), Bolker, Guillemin, and Holm in [BGH, Section 2] state that:"... the star of a vertex (of a graph) is a combinatorial analogue of the tangent space to a manifold at a point ...".

We observe that this is in complete agreement with the point of view of our introductory sections. Indeed, a "tangent space" is in all exhibited examples either an object in $Ob(\mathcal{G})$ or alternatively the fibre of a tautological bundle over the associated groupoid $\mathcal{G}$.

3.3 Holonomy vs. NaCl-invariant of a cubical complex

In this section we apply the ideas outlined in earlier sections to the case of cubical complexes.

Recall that a cell complex K is cubical if it is a regular CW-complex such that the associated face poset P_K is cubical in the sense of the following definition.

Definition 3.2. *P is a cubical poset if:*

(a) *for each $x \in P$, the subposet $P_{\leq x}$ is isomorphic to the face poset of some cube I^q;*

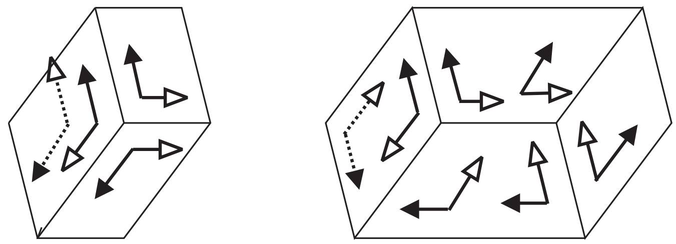
(b) *P is a semilattice in the sense that if a pair $x, y \in P$ is bounded from above then it has the least upper bound.*

If a space X comes equipped with a standard cubification, clear from the context, this cubical complex is denoted by $\{X\}$, the associated k-skeleton is denoted by $\{X\}_{(k)}$ etc. For example $\{I^d\}_{(k)}$ is the k-skeleton of the standard cubification of the d-cube, similarly $\{\mathbb{R}^d\}_{(k)}$ is the k-skeleton of the standard cubification of $\mathbb{R}^d$ associated to the lattice $\mathbb{Z}^d$.

The group B_k of all symmetries of a k-cube is isomorphic to the group of all signed, permutation $(k \times k)$-matrices. Its subgroup of all matrices with even number of (-1)-entries is denoted by B_k^{even}. The vertex-edge graph of a k-cube is well known to be bipartite, i.e. colorable with two colors (Figure 8) and B_k^{even} can be described as the set of all elements in B_k that preserve this coloring.

Given a (pure) d-dimensional cubical complex X, the associated groupoid is denoted by $\mathcal{C}(X)$. As a cubical counterpart of Joswig's groupoid, it was introduced in [Ž05b] [Ž05] and applied to problems related to embeddings of cubical complexes into cubical lattices (problem of S. Novikov, Figure 1 (b)) and questions of "bubble modifications" of cubical complexes (problem of N. Habbeger, Figure 2 (d)).

Both applications were based on a holonomy type, $\mathbb{Z}_2$-invariant $I(K)$ of a cubical complex K introduced in [Ž05].



(A) (B)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Figure 7: Examples of holonomies in a 2-dimensional cubical lattice.

Definition 3.3. *Suppose that K is a k-dimensional cubical complex and let $\Pi(K,\sigma)$ be its combinatorial holonomy group based at $\sigma \in K$. By definition let $I(K) = 0$ if $\Pi(K,\sigma) \subset B_k^{even}$ for all σ, and $I(K) = 1$ in the opposite case.*

The reader is referred to [Ž05] for more detailed exposition of results related to this invariant. Following a suggestion of G. Ziegler, we give a useful criterion which in many cases of interest enables us to prove that $I(K) = 0$.

Proposition 3.4. *(NaCl-criterion) $I(K) = 0$ if the cubical complex K can be colored with two colors such that adjacent vertices are always of different colors, equivalently if the vertex-edge graph of K is bipartite.*

Proof: (outline) If such a coloring exists then it is preserved by the parallel transport in the groupoid $\mathcal{C}(K)$. □

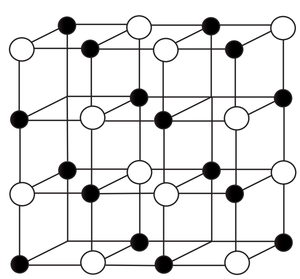


Figure 8: Sodium chloride NaCl as a cubical complex.

Let us define the "NaCl-invariant" of a cubical complex K by the requirement that $\mathrm{NaCl}(K)$ is 0 (respectively 1) if its vertex-edge graph can (cannot) be colored by 2 colors (Figure 8). Proposition 3.4 says that $I(K) \leq \mathrm{NaCl}(K)$. It is not difficult to find examples of complexes which prove that in general $I(K) \neq \mathrm{NaCl}(K)$.

Example 3.5. Indeed, let K be a cubical complex such that $\mathrm{NaCl}(K) = 0$. Let us identify two vertices u and v, non-adjacent in K, which are nevertheless

assigned different colors. Let K' be the cubical complex $K' := K/u \approx v$. This identification does not effect the holonomy group of K i.e. $\Pi_K \cong \Pi_{K'} \subset B_k^{even}$, hence $I(K) = I(K') = 0$. On the other hand $\mathrm{NaCl}(K') = 1$.

Let us clarify the relationship between invariants $I(K)$ and $\mathrm{NaCl}(K)$, at least in the important case of cubifications of manifolds. As it was kindly pointed by the referee, the following proposition, modelled on Proposition 6 from [J01], provides a fairly complete and natural answer to this question.

Proposition 3.6. *Suppose that K is a cubical complex which is globally and locally (strongly) connected. This means that both the groupoid $\mathcal{C}(K)$ and each of its subgroupoids $\mathcal{C}(\mathrm{Star}(v))$ are connected where v is a vertex in K. Then*

$$I(K) = \mathrm{NaCl}(K).$$

Proof: The proof is similar to the proof of Proposition 6 from [J01]. Since $I(K) \leq \mathrm{NaCl}(K)$, it is sufficient to show that if $I(K) = 0$ then the vertices of K can be colored with two colors such that no two vertices with same color are adjacent, i.e. that the vertex-edge graph of K is bipartite.

The required coloring of vertices of K is defined as follows. Select a top dimensional cell $c_0 \in K$ and color its vertices with two colors. Each other cell $c \in K$ is (strongly) connected with c_0, i.e. connected in the sense of the groupoid $\mathcal{C}(K)$. Given a "path" (morphism) θ in $\mathcal{C}(K)$, connecting c_0 and c, the coloring of c_0 can be extended along this path to a coloring of c. By assumption $\Pi(\mathcal{C}(K)) = 0$, hence this coloring of c does not depend of the path θ. On the other hand a vertex $v \in c$ might receive a different color from another cell c' such that $v \in c'$. By assumption $\mathrm{Star}(v)$ is also (strongly) connected which guaranties that there exists a path from c to c' inside $\mathrm{Star}(v)$. This guarantees that this is not possible which concludes the proof of the proposition. □

The following result, illustrates the elegance of the NaCl-approach. It shows that Theorem 3.2 from [Ž05] is really a relative of results from [J01] about colorings of simple polytopes.

Theorem 3.7. *Suppose that K is a k-dimensional cubical complex which is embeddable/mappable to $\{Z\}_{(k)}$, the k-dimensional skeleton of the standard cubical decomposition of a generic zonotope $Z = [-v_1, v_1] + \ldots + [-v_n, v_n]$. Then $I(K) = 0$.*

Proof: Each vertex $w = \epsilon_1 v_1 + \ldots + \epsilon_n v_n$ of Z is colored in "black" (respectively'"white") if there is an odd (even) number of occurrences of -1 in the sequence $\epsilon_1, \ldots, \epsilon_n$. It is not difficult to show that in this coloring no two adjacent vertices are colored by the same color. This implies that $\mathrm{NaCl}(K) = 0$ for each k-dimensional subcomplex of Z. □

3.4 Generalized Lovász conjecture

One of the central problems of topological graph theory is to explore how the topological complexity of a graph complex $X(G)$ is reflected in the combinatorial complexity of the graph G itself. The results one is often interested in come in the form of implications

$$\alpha(X(G)) \geq p \Rightarrow \xi(G) \geq q,$$

where $\alpha(X(G))$ is a topological invariant of the complex $X(G)$, while $\xi(G)$ is a combinatorial invariant of the graph G. The earliest statement of this type is the celebrated result of L. Lovász which is today often formulated in the form of an implication

$$Hom(K_2, G) \text{ is } k\text{-connected } \Rightarrow \chi(G) \geq k+3,$$

where $Hom(K_2, G)$ is one of many (essentially equivalent) $\mathbb{Z}_2$-complexes associated to G, see [Ko] and [M03] as overviews and guides to the literature. $Hom(K_2, G)$ is a special case of a general graph complex $Hom(H, G)$ (also introduced by L. Lovász), a cell complex which functorially depends on the input graphs H and G.

An outstanding conjecture in this area, referred to as the "Lovász conjecture", was that one obtains a better bound if the graph K_2 is replaced by an odd cycle C_{2r+1}. More precisely Lovász conjectured that

$$Hom(C_{2r+1}, G) \text{ is } k\text{-connected } \Rightarrow \chi(G) \geq k+4. \tag{3}$$

This conjecture was confirmed in [BK04], see also [Ko] for a more detailed account.

The main observation of [Ž05b] was that the general theory of groupoids, in particular the Joswig groupoid $\mathcal{J}(K)$, provide a deep insight into the Lóvasz conjecture and its ramifications. As a consequence one obtains the implication

$$Hom(\Gamma, K) \text{ is } k\text{-connected } \Rightarrow \chi(K) \geq k+d+3$$

which under suitable assumption on the "test complex" Γ and the assumption that integer k is odd, extends the result of Babson and Kozlov to the case of pure d-dimensional simplicial complexes. Moreover, this approach yields a short and conceptual proof of the Lovász conjecture for k odd. The reader is referred to [Ž05] for an exposition of these and related results.

Subsequently the approach based on groupoids was extended and incorporated into equivariant index theory by C. Schultz [S05] [S06] who developed new powerful methods leading to deep understanding of Hom-complexes and further analogues of (3).

There are two new, short and elegant, proofs of the Babson-Kozlov-Lovász theorem. The proof in [S06] is based on the evaluation of the cohomological $\mathbb{Z}_2$-index while the more recent proof [Ko06] relies on a combinatorial evaluation of the height of the associated Stiefel-Whitney characteristic class.

3.4.1 The main observation

In this section we briefly describe the nature of the "mathematical revelation" that pointed to the connection between [BK04] and [J01], led to [Ž05b] and [Ž05], and served as the author's main initial motivation for starting the program of studying combinatorial groupoids.

It is well known that a graph $G = (V_G, E_G)$ admits a coloring with not more than m colors if and only there exists a graph homomorphism $c : G \to K_m$ from G to the complete graph with m vertices (Example 2.6).

Given graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, the associated graph complex $Hom(G, H)$ is the cell complex where each cell is indexed by a multivalued function $\eta : V_G \to 2^{V_H} \setminus \{\emptyset\}$ such that if $(i, j) \in E_G$ then for each $\alpha \in \eta(i)$ and each $\beta \in \eta(j)$, $(\alpha, \beta) \in E_H$, [Ko] [BK04] [S06]. For example it is a well known fact that the Hom-complex $Hom(K_2, K_m)$ between complete graphs K_2 and K_m is homeomorphic to a $(m-2)$-dimensional sphere.

A graph G (without loops and multiple edges) can be interpreted as a 1-dimensional simplicial complex. Let $\mathcal{J}(G)$ be the corresponding Joswig's groupoid, Example 2.4. Each edge $e \in E_G$ itself can be interpreted as a subgraph of G isomorphic to K_2. The map $\Gamma_G : E_G \to Top$ defined by

$$e \mapsto Hom(e, K_m)$$

is a spherical bundle over the set E_G of edges of G in the sense of Section 2.1. There is a "forgetful" continuous map $\phi_e : Hom(G, K_m) \to Hom(e, K_m)$ for each edge $e \in E_G$.

The key observation: The "parallel transport" with respect to the Joswig's groupoid $\mathcal{J}(G)$ preserves the homotopy type of the map ϕ_e. If $G \cong C_{2r+1}$ is an odd cycle, then the holonomy group $\Pi(\mathcal{J}(G)) \cong \mathbb{Z}_2$ is nontrivial and as a consequence there is a homotopy equivalence

$$\phi_e \simeq \phi_e \circ \alpha_e \tag{4}$$

where $\alpha_e \in \Pi_e$ is the nontrivial element of the corresponding holonomy group.

The homotopy (4) has cohomological consequences which eventually, in light of the naturality of Hom-construction, can be used to show that a coloring $c : G \to K_m$ is not possible. The details of this construction and its ramifications can be found in [Ž05], see also [Ž05b] for a preliminary version.

3.5 Afterword

There are other examples of applications of discrete connections, discrete holonomies (combinatorial groupoids) etc. that have not been covered by this review. A notable example is the paper of Novikov [N04], see also Novikov and Dynnikov [DN02], and the references in these papers.

Novikov and his followers have studied discrete connections on triangulated manifolds as a part of a general programme of developing discretized differential geometry, finding discrete analogs of important differential operators, describing discrete analogs of complete integrable systems etc.

These developments are naturally linked with the “Discrete differential geometry” in the sense of Bobenko and Suris [BS], a broad new area where differential geometry of smooth curves, surfaces and other manifolds interacts with discrete geometry, using tools and ideas from all parts of mathematics, and having applications ranging from integrable dynamical systems to computer graphics.

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References

[BBLL] E. Babson, H. Barcelo, M. De Longueville, and R. Laubenbacher, *Homotopy theory for graphs*, arXiv:math.CO/0403146 v1 9 Mar 2004, Journal of Algebraic Combinatorics, in press.

[BC] E.K. Babson and C. Chan, *Counting faces for cubical spheres modulo two*, Discrete Math. 212 (2000), 169–183.

[BK03] E. Babson, D.N. Kozlov, *Complexes of graph homomorphisms*, arXiv:math.CO/0310056 v1 5 Oct 2003, to appear in Israel J. Math.

[BK04] E. Babson, D.N. Kozlov, *Proof of the Lovász conjecture*, Annals of Mathematics, Accepted papers (2005), arXiv:math.CO/0402395 v2, 2004.

[BL] H. Barcelo and R. Laubenbacher, *Perspectives on A-homotopy theory and its applications*, preprint.

[BKLW] H. Barcelo, X. Kramer, R. Laubenbacher, and C. Weaver, *Faundations of a connectivity theory for simplicial complexes*, preprint.

[BEE] M.W. Bern, D. Eppstein, and J.G. Erikson, *Flipping cubical meshes*, Engineering with Computers 18 (2002), 173–187.

[Bj95] A. Björner, *Topological methods.* In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, North-Holland, Amsterdam, 1995.

[BVSWZ] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, *Oriented Matroids*, Encyclopedia of Mathematics and its Applications **46**, Cambridge University Press 1993.

[BS] A.I. Bobenko, Yu.B. Suris, *Discrete Differential Geometry; Consistency as Integrability*, arXiv:math.DG/0504358 v1 Apr 2005.

[BGH] E.D. Bolker, V.W. Guillemin, T.S. Holm, *How is a graph like a manifold?*, arXiv:math.CO/0206103 v1 Jun 2002.

[Br87] R. Brown, *From groups to groupoids: a brief surway*, Bull. London Math. Soc. 19(1987) 113–134.

[Br97] R. Brown, *Groupoids and crossed objects in algebraic topology*, Homology, Homotopy and Applications, Vol. 1, 1999, No. 1, pp 1–78.

[Br] R. Brown, *Topology and Groupoids*, Booksurge LLC 2006; retitled, revised, updated and extended edition of *Topology: A Geometric Account of General Topology, Homotopy Types and the Fundamental Groupoid*, Ellis Horwood Limited 1988, and of *Elements of Modern Topology*, McGraw-Hill 1968.

[BP00] V.M. Buchstaber, T.E.Panov, *Torus actions, combinatorial topology, and homological algebra*, Russian Math. Surveys 55 (2000), no. 5, 825–921, arXiv:math.AT/0010073.

[BP02] V.M. Buchstaber, T.E.Panov, *Torus actions and their applications in topology and combinatorics*, Univ. Lecture Ser., A.M.S. 2002.

[CF60] P.E. Conner, E.E. Floyd, *Fixed point free involutions and equivariant maps*, Bull. Amer. Math. Soc. **66** (1960), 416–441.

[C95] A. Connes, *Non commutative Geometry*, Academic Press 1995.

[CrRy] H. Crapo, J. Ryan, *Spacial realization of linear scenes*, Structural Topology 13 (1986), 33–68.

[DSS86] N.P. Dolbilin, M.A. Shtan'ko, and M.I.Shtogrin, *Cubic subcomplexes in regular lattices*, Dokl. Akad. Nauk., SSSR 291(1986), English translation: *Soviet Math. Dokl.*, 34(1987).

[DSS87] N.P. Dolbilin, M.A. Shtan'ko, and M.I.Shtogrin. *Cubic manifolds in lattices*, Izv. Ross. Akad. Nauk., Ser. Mat. 58(1994), 93–107, English translation: *Russ. Acad. Sci. Izv. Math.*, 44 (1995), 301–313.

[DN02] I.A. Dynnikov, S.P. Novikov, *Geometry of the triangle equation on two-manifolds*, arXiv:math-ph/0208041 v2 Oct 2002.

[Epp99] D. Eppstein, *Linear complexity hexaedral mesh generation*, Comput. Geom. 12 (1999), 3–16.

[Fu99] L. Funar, *Cubulations, immersions, mappability and a problem of Habegger*, Ann. Sci. E.N.S. 32 (1999), 681–700.

[Fu99b] L. Funar, *Cubulations mod bubble moves*, in Proc. Conf. Low Dimensional Topology, Funchal, Madeira 1998 (H. Nencka, Ed.) *Contemporary Math.* 233, 29–43, A.M.S. 1999.

[Fu05] L. Funar, *Surface quadrangulations mod flips*, preprint (January, 2005) available at `http://www-fourier.ujf-grenoble.fr/~funar`.

[Ga04] A.A. Gaifulin, *Local formulae for combinatorial Pontriagin classes*, Izvestya RAN: Ser. Mat. 68:5 13–66.

[GKM] M. Goresky, R. Kottwitz, R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. 131 (1998), 25–83.

[H71] P.J. Higgins, *Categories and groupoids*, Van Nostrand Reinhold Co., London 1971.

[I01] I. Izmestiev, 3*-dimensional manifolds defined by simple polytopes with colored facets*, Mat. Zametki **69** (2001), 375–382, (English translation: Math. Notes **69**, 1987).

[I01b] I. Izmestiev, *Free actions of a torus on a manifold* $\mathcal{L}_P$ *and the projectivity group of a polytope* P, Uspekhi Mat. Nauk, **56** (2001) pp. 169–170, English translation: Russian Math. Survays **56** (2001).

[IJ02] I. Izmestiev and M. Joswig, *Branched coverings, triangulations, and* 3*-manifolds*, Adv. Geom. **3** (2003), 191–225, arXiv:math.GT/0108202 v2 20 Mar 2002.

[J01] M. Joswig, *Projectives in simplicial complexes and coloring of simple polytopes*, Math. Z. **240** (2002) no. 2, 243–259, arXiv:math.CO/0102186 v3 27 Jun 2001.

[J01b] M. Joswig, *Projectives in simplicial complexes and coloring of simple polytopes* (extended abstract), Russ. Math. Surv. **56** (2001), 584–585.

[K91] O.R. Karalashvili, *On mappings of cubic manifolds into the standard lattice of Euclidean space*, Trudy Mat. Inst. Steklov 196(1991), 86–89, translated in: Proc. Steklov Inst. Math. 196(1992).

[K95] R. Kirby, *Problems in low-dimensional topology*, in "Geometric Topology", Georgia International Topology Conference, (W.H. Kazez, Editor), AMS-IP Studies in Advanced Math., **2**, part 2, 35–473, 1995.

[Ko99] D.N. Kozlov, *Complexes of directed trees*, J. Combin. Theory Ser. A **88** (1999), no. 1, 112–122.

[Ko04] D.N. Kozlov, *A simple proof for folds on both sides in complexes of graph homomorphisms*, arXiv:math.CO/0408262 v2 Dec 2004.

[Ko] D.N. Kozlov, *Chromatic numbers, morphism complexes, and Stiefel-Whitney characteristic classes*, "Geometric Combinatorics", IAS/Park City Mathematics Series 14, American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ.

[Ko06] D.N. Kozlov, *Cobounding odd cycle colorings*, arXiv:math.AT/0602561 Feb 2006.

[L78] L. Lovász, *Kneser's conjecture, chromatic number and homotopy*, J. Comb. Theory, Ser. A, **25**:319–324, 1978.

[M03] J. Matoušek, *Using the Borsuk-Ulam Theorem; Lectures on Topological Methods in Combinatorics and Geometry*, Springer Universitext, Berlin 2003.

[N96] S.P. Novikov, *Topology I*, Encyclopaedia Math. Sci. **12**, Springer-Verlag, Berlin 1996.

[N04] S.P. Novikov, *Discrete connections on the triangulated manifolds and difference linear equations*, arXiv:math-ph/0303035 v2 Apr 2004.

[Pe] R. Penrose, *On the cohomology of impossible figures*, Structural Topology 17 (1991), 11–16.

[SZ04] A. Schwartz, G.M. Ziegler, *Construction techniques for cubical complexes, odd cubical* 4*-polytopes, and prescribed dual manifolds*, Experimental Math., 13:385413, 2004, arXiv:math.CO/0310269 v3 2 Jan 2004.

[S05] C. Schultz, *A short proof of* $w_1^n(Hom(C_{2r+1}, K_{n+2})) = 0$ *for all* n *and a graph colouring theorem of Babson and Kozlov*, arXiv:math.AT/0507346 v2 Aug 2005.

[S06] C. Schultz, *Graph colorings, spaces of edges and spaces of circuits*, preprint 2006.

[W96] A. Weinstein, *Groupoids: Unifying Internal and External Symmetry*, Notices of the A.M.S., vol. **43**, July 1996.

[Z92] G.M. Ziegler, *Some "embedding problems"*, unpublished manuscript December 1992.

[Z98] G.M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics 152, Springer 1995, 2nd ed. 1998.

[Ž04] R. Živaljević, *Topological methods*, in CRC Handbook of Discrete and Computational Geometry (new edition), J.E. Goodman, J. O'Rourke (eds.), Boca Raton 2004.

[Ž05] R. Živaljević, *Combinatorial groupoids, cubical complexes, and the Lovász conjecture*, arXiv:math.CO/0510204 v2 Oct 2005.

[Ž05b] R. Živaljević, *Parallel transport of Hom-complexes and the Lovász conjecture*, arXiv:math.CO/0506075 v1 3 Jun 2005.